§ Diagonalizability.

Recall: Matrix A is called diagonalizable if A is similar to a diagonal matrix.

i.e., $\exists$ invertible matrix Q, s.t. $Q^{-1}AQ$ is $\begin{pmatrix} \ddots & 0 \\ 0 & \ddots \end{pmatrix}$.

Question: Is every matrix diagonalizable?

Answer: No!!! e.g. $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, $\begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix}$

Def: Suppose $T \in \mathcal{L}(V)$. V finite-dim.

Then T is diagonalizable if one of the following equivalent conditions is true.

(1) (Vector Space form) V has an ordered basis β in which each basis vector is an eigenvector of T.

(2). (Matrix Form) V has an ordered basis β s.t. $[T]_\beta$ is diagonal.

pf of equivalence. Suppose $\beta = \{\vec{v_1}, \cdots, \vec{v_n}\}$

$$T(\vec{v_j}) = \lambda_j \cdot \vec{v_j} \iff [T]_\beta = \begin{pmatrix} \lambda_1 & 0 & & 0 \\ 0 & \lambda_2 & & \\ \vdots & \vdots & \ddots & \\ 0 & 0 & & \lambda_n \end{pmatrix}$$

Prop: $T = L_A : F^n \to F^n$ is diagonalizable (as an operator)

if and only if A is diagonalizable (as a matrix)

i.e., $\exists Q$ s.t. $Q^{-1} A Q$ is diagonal.

Pf: Let γ standard basis, then $[T]_\gamma = A$

Change of basis formula $\Rightarrow$ $[T]_\beta = Q^{-1} \cdot \underbrace{[T]_\gamma}_{= A} \cdot Q$

Note: NOT all linear operator are diagonalizable:

e.g. T rotation (by 90°) on $\mathbb{R}^2$.

We know T has no eigenvalue & eigenvector.

Use Vector Space Form of diagonalizability $\Rightarrow$ T is not diag.

Consequently, not all matrices are diagonalizable!

e.g. $[T]_\beta = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ is not diagonalizable.

☆ Question: When is T diagonalizable?

(1) Simple Criterion · in terms of characteristic polynomial $f_T(t)$ only.

· normal operator Later ...

(2) General Necessary & Sufficient Condition

(involves additional data than char poly — harder to check!)

(not necessarily distinct)

Def: A poly $f(t) \in P(F)$ splits over F if $\exists c \; \& \; a_1 \cdots a_n \in F$

s.t. $f(t) = c(t-a_1)\cdots(t-a_n)$, i.e., has n roots

Ex:

- If $F = \mathbb{R}$. Not all $f \in P(\mathbb{R})$ can split over $\mathbb{R}$, e.g. $f(t) = t^2+1$
- If $F = \mathbb{C}$, then any $f \in P(\mathbb{C})$ splits over $\mathbb{C}$ (by Fund Thm of Alg.)

Theorem: The characteristic poly of a diagonalizable linear operator on a finite-dim space V splits over F.

(Necessary condition)

Pf: If V is n-dim and $T \in \mathcal{L}(V)$ is diagonalizable.

Then $\exists$ a basis $\beta \subset V$. s.t. $[T]_\beta = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}$

Then. $f_T(t) = \det\left([T]_\beta - tI_n\right) = \begin{pmatrix} \lambda_1 - t & & & 0 \\ & \lambda_2 - t & & \\ & & \ddots & \\ 0 & & & \lambda_n - t \end{pmatrix}$

$= (-1)^n (t-\lambda_1) \cdots (t-\lambda_n)$

$\square$

★ **Theorem.** Let $T \in \mathcal{L}(V)$ with $\dim V = n$.
(Sufficient Cond.)
If T has n distinct eigenvalues, then T is diagonalizable.

Pf: Let $\lambda_1, \cdots, \lambda_n$ are n distinct eigenvalues of T.

For each λ_i, let $\vec{v_i}$ be the associated eigenvectors

Define $\beta = \{\vec{v_1}, \cdots, \vec{v_n}\}$

Claim. β is linearly indep. (pf below).

$\Rightarrow$ β is a basis for V. $\Rightarrow$ T is diagonalizable $\square$

Lemma: A set of eigenvectors associated to distinct eigenvalues of T are linearly indep.

Pf: Induction on $k :=$ # of vectors.

- $k=1$. $\{\vec{v}_1\}$ eigenvector $\neq \vec{0}$ is lin. indep.
- Assume true for k vectors $\{\vec{v}_1, \cdots, \vec{v}_k\}$. Add the vector $\vec{v}_{k+1}$.

Let $S = \{\vec{v}_1, \cdots, \vec{v}_{k+1}\}$ where $T(\vec{v}_i) = \lambda_i \vec{v}_i$ and $\lambda_1, \cdots, \lambda_{k+1}$ distinct.

To show S lin. indep. let $\boxed{\sum_{i=1}^{k+1} a_i \vec{v}_i = 0.}$ (1)

Apply linear operator T: $\quad T\left(\sum_{i=1}^{k+1} a_i \vec{v_i}\right) = 0$

$$\Rightarrow \boxed{\sum_{i=1}^{k+1} a_i \lambda_i \vec{V_i} = 0} \qquad (2)$$

(2) $-\lambda_{k+1} \cdot$ (1): $\quad \sum_{i=1}^{k} a_i (\lambda_i - \lambda_{k+1}) \vec{V_i} = 0$

By Inductive hyp, $\{\vec{V_1}, \cdots, \vec{V_k}\}$ lin. indep $\Rightarrow$ $a_i \underbrace{(\lambda_i - \lambda_k)}_{\neq 0} = 0$

Since $\lambda_1, \cdots, \lambda_k$ distinct $\Rightarrow \lambda_i - \lambda_k \neq 0 \Rightarrow a_1 = \cdots = a_k = 0$.

(1) $\Rightarrow a_{k+1} \vec{V_{k+1}} \neq 0 \Rightarrow a_{k+1} = 0$; hence $\{\vec{V_1}, \cdots, \vec{V_{k+1}}\}$ lin. indep. $\square$

Summary

- **Necessary Condition**: If T is diagonalizable, then $f_T(t)$ must split. i.e., $f_T(t) = c(t-a_1)\cdots(t-a_n)$

- **Sufficient Condition**: If $f_T(t)$ has n distinct roots. i.e., $a_1, \cdots, a_n$ are distinct then T is diagonalizable

<u>Example:</u> $T = L_A : \mathbb{F}^2 \longrightarrow \mathbb{F}^2$. $A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$

Char poly: $f_T(t) = t^2 + 1$.

← geometrically rotation.

- When $\mathbb{F} = \mathbb{R}$. $f_T(t)$ does not split $\Rightarrow$ T is NOT diagonalizable.
- When $\mathbb{F} = \mathbb{C}$ $f_T(t) = (t+i)(t-i)$, 2 distinct roots $\Rightarrow$ T is diagonalizable.

$$\left(\text{Indeed, } \quad T\begin{pmatrix} 1 \\ -i \end{pmatrix} = i\begin{pmatrix} 1 \\ -i \end{pmatrix} \qquad T \cdot \begin{pmatrix} 1 \\ i \end{pmatrix} = -i\begin{pmatrix} 1 \\ i \end{pmatrix} \right.$$

$$\left. \Rightarrow \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} = Q^{-1} \cdot \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \cdot Q, \text{ where } Q = \begin{pmatrix} 1 & 1 \\ -i & i \end{pmatrix}. \quad Q^{-1} = \frac{1}{2}\begin{pmatrix} 1 & i \\ 1 & -i \end{pmatrix} \right)$$

Def: Let λ be an eigenvalue of $T \in \mathcal{L}(V)$ with char poly $f_T(t)$.

The algebraic multiplicity of λ, denoted $\mu_T(\lambda)$,

is the multiplicity of λ as a zero of $f(t)$

i.e., the largest k, s.t. $(t-\lambda)^k \mid f_T(t)$

Note: When char. poly $f_T(t)$ splits, write $f_T(t) = c(t-\lambda_1)^{m_1}(t-\lambda_2)^{m_2}\cdots(t-\lambda_k)^{m_k}$

Then m_i = algebraic multiplicity of λ_i. $\lambda_1, \cdots, \lambda_k$ distinct

Also. $m_1 + \cdots + m_k = n = \dim V$.

Example : • 1 is an eigenvalue of $I_V : V \to V$.

$$\begin{pmatrix} 1-t & & 0 \\ & \ddots & \\ 0 & & 1-t \end{pmatrix}$$

with $\mu_{I_V}(1) = \dim V$.

$$f_{I_V}(t) = (1-t)^n$$

• For $A = \begin{pmatrix} 3-t & 1 & 0 \\ 0 & 3-t & 4 \\ 0 & 0 & 4-t \end{pmatrix}$

$$f_A(t) = (3-t)^2 \cdot (4-t)$$

$$\Rightarrow \mu_A(3) = 2. \qquad \mu_A(4) = 1.$$

Recall: Eigenspace $E_\lambda := N(T-\lambda I_V) = \{ \vec{x} \in V : T(\vec{x}) = \lambda \vec{x} \}$

Def: $\gamma_T(\lambda) := \dim E_\lambda$. the geometric multiplicity of λ.

Prop: Let $T \in \mathcal{L}(V)$, V finite-dim space. Let λ be an eigenvalue of T

Then $1 \le \gamma_T(\lambda) \le \mu_T(\lambda)$.

Pf: • Take a basis $\{\vec{v_1}, \dots, \vec{v_p}\}$ for E_λ and extend to a basis $\beta = \{\vec{v_1}, \dots, \vec{v_p}, \vec{v}_{p+1}, \dots, \vec{v_n}\}$ for V.

Then $$[T]_\beta = \left(\begin{array}{cccc|c} \lambda & 0 & \cdots & 0 & ** \\ 0 & \lambda & & \vdots & \\ \vdots & 0 & & \vdots & ** \\ \vdots & \vdots & \ddots & \lambda & \\ \hline \vdots & \vdots & & 0 & \\ 0 & 0 & \cdots & 0 & ** \end{array}\right) = \left(\begin{array}{c|c} \lambda I_p & B \\ \hline O & C \end{array}\right) \quad - tI_n$$

$$\Rightarrow \quad f_T(t) = \det \left(\begin{array}{c|c} (\lambda - t) I_p & B \\ \hline 0 & C - t I_{n-p} \end{array} \right)$$

$$= (\lambda - t)^p \cdot \det(C - t I_{n-p})$$

So $\mu_T(\lambda) \geq p = \gamma_T(\lambda)$

- On the other hand, λ eigenvalue $\Leftrightarrow$ $T - \lambda I_V$ is not invertible $\Leftrightarrow$ $N(T - \lambda I_V) = E_\lambda$ has $\dim \geq 1$.

Hence. $1 \leq \gamma_T(\lambda)$. $\square$

Theorem: (Sufficient & Necessary Condition for diagonalizability)

Suppose $T \in \mathcal{L}(V)$. s.t. the char poly $f_T(t)$ splits. $\mu_T(\lambda_1)+\cdots+\mu_T(\lambda_k) = n = \dim V$

Let $\lambda_1, \cdots, \lambda_k$ be the distinct eigenvalues of T.

Then:

$\Rightarrow \gamma_T(\lambda_1)+\cdots+\gamma_T(\lambda_k) = n$

(a) (test) T is diagonalizable iff $\boxed{\mu_T(\lambda_i) = \gamma_T(\lambda_i)}$ $\forall i = 1, \cdots, k$.

(b). If T is diagonalizable and β_i is a basis for E_{λ_i} $\forall i$.

(explicit basis) then $\beta := \beta_1 \cup \cdots \cup \beta_k$ is a basis for V.

Consisting of eigenvectors of T (so that $[T]_\beta$ is diagonal)

Example

(1) For $A = \begin{pmatrix} 3 & 1 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{pmatrix}$. $f_A(t) = (3-t)^2 \cdot (4-t)$. splits over $\mathbb{R}$.

$\mu_A(3) = 2 \quad \mu_A(4) = 1$.

- Since $1 \leq \gamma_A(4) \leq \mu_A(4) \Rightarrow \gamma_A(4) = 1 = \mu_A(4)$.

- How about $\gamma_A(3)$?

$$E_3 = N(A - 3I) = N\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \Rightarrow \gamma_A(3) = 1 < \mu_A(3)$$

$$= \left\{ c\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \right\} \quad \text{1-dim.} \qquad \Rightarrow A \text{ is not diagonalizable.}$$

(2). Consider $T: P_2(\mathbb{R}) \longrightarrow P_2(\mathbb{R})$ defined by

$$f(x) \rightsquigarrow f(1) + f'(0)\cdot x + \left(f'(0) + f''(0)\right)\cdot x^2$$

Let α be the standard basis for $P_2(\mathbb{R})$. $\alpha = \{1, x, x^2\}$

Then $[T]_\alpha = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & 2 \end{pmatrix}$

$1 \rightsquigarrow 1$

$x \rightsquigarrow 1 + x + x^2$

$x^2 \rightsquigarrow 1 + 0\cdot x + 2x^2$.

$\Rightarrow f_T(t) = (1-t)^2\cdot(2-t)$ Splits over $\mathbb{R}$.

and the eigenvalues of T are 1 and 2

$\mu_T(1) = 2 \qquad \mu_T(2) = 1$

Then determine the geometric mult :

- We have $\gamma_T(2) = 1 = \mu_T(2)$

- Note that $[T - 1 \cdot I_V]_\alpha = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix} \Rightarrow \text{rank} = 1$

$$\Rightarrow \gamma_T(1) = \dim N(T - I_V) = 2 = \mu_T(1)$$

So T is diagonalizable

Indeed, for $[T]_\alpha$, the eigenspaces are.

$E_1 = \left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \in \mathbb{R}^3 : x_2 + x_3 = 0 \right\}$, so $\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} \right\}$ is a basis.

$E_2 = \left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \in \mathbb{R}^3 : x_2 = -x_1 + x_2 + x_3 = 0 \right\}$, so $\left\{ \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \right\}$ is a basis.

$\Rightarrow \gamma = \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \right\}$ is a basis for $\mathbb{R}^3$ consisting of eigenvectors of $[T]_\alpha$.

and it corresponds to the basis $\beta = \{1, x - x^2, 1 + x^2\}$ eigenvectors of T.

$\square$

Lemma: Suppose $T: V \longrightarrow V$ has distinct eigenvalues $\lambda_1, \cdots, \lambda_k$.

For each $i = 1, \cdots, k$, let $S_i \subset E_{\lambda_i}$ be a finite lin. indep subset

Then $S = S_1 \cup S_2 \cup \cdots \cup S_k$ is a linearly indep subset of V.

Recall: "A set of eigenvectors associated to distinct eigenvalues of T

are linearly indep." $(*)$

Pf: Suppose $S_1 = \{ \vec{v}_{11}, \vec{v}_{12} \; \cdots \; , \vec{v}_{1n_1} \}$ $\quad \vec{w}_1 = a_{11}\vec{v}_{11} + \cdots + a_{1n_1}\vec{v}_{1n_1} \in E_{\lambda_1}$

$S_2 = \{ \vec{v}_{21}, \vec{v}_{22} \; \cdots \; \vec{v}_{2n_2} \}$ $\quad \vec{w}_2 = a_{21}\vec{v}_{21} + \cdots + a_{2n_2}\vec{v}_{2n_2} \in E_{\lambda_2}$

$\cdots \qquad\qquad \cdots$

$S_k = \{ \vec{v}_{k1}, \vec{v}_{k2} \; \cdots \; \vec{v}_{kn_k} \}$ $\quad \vec{w}_k = a_{k1}\vec{v}_{k1} + \cdots + a_{kn_k}\vec{v}_{kn_k} \in E_{\lambda_k}$

then $S = \{ \vec{v}_{ij} : 1 \le i \le k, 1 \le j \le n_i \}$

Assume $$\sum_{i=1}^{k} \underbrace{\sum_{j=1}^{n_i} a_{ij} \vec{v}_{ij}}_{\vec{w}_i} = 0$$

Let $\vec{w_i} = \sum_{j=1}^{n_i} a_{ij} \vec{v_{ij}} \in E_{\lambda_i}$; Then $\sum_{i=1}^{k} \vec{w_i} = 0$

Then (*) implies $\vec{w_i} = 0 \quad \forall i$

Hence $\sum_{j=1}^{n_i} a_{ij} \vec{v_{ij}} = 0$. As S_i lin indep, $a_{ij} = 0$

Hence $S = S_1 \cup \dots \cup S_k$ lin. indep.

$\square$

Pf of Thm:

"$\Rightarrow$". Assume T is diagonalizable, then there's a basis β of V consisting of eigenvectors of T.

$$[T]_\beta = \begin{pmatrix} \lambda_1 & & & & & & & & \\ & \ddots & & & & & & & \\ & & \lambda_1 & & & & & & \\ & & & \lambda_2 & & & & & \\ & & & & \ddots & & & & \\ & & & & & \lambda_2 & & & \\ & & & & & & \lambda_k & & \\ & & & & & & & \ddots & \\ & & & & & & & & \lambda_k \end{pmatrix}$$

Let $\beta_i = \beta \cap E_{\lambda_i}$ $(1 \le i \le k)$

Then $\#\beta_i \le \dim E_{\lambda_i} = \gamma(\lambda_i) \le \mu(\lambda_i)$.

But $\dim V = \#\beta = \sum_{i=1}^{k} \#\beta_i \le \sum_{i=1}^{k} \mu(\lambda_i) = \dim V$

$\Rightarrow \#\beta_i = \gamma(\lambda_i) = \mu(\lambda_i)$

"$\Leftarrow$": Assume $\gamma(\lambda_i) = \mu(\lambda_i)$ $\forall i = 1, \cdots, k$.

Let β_i be a basis for E_{λ_i}. Let $\beta = \beta_1 \cup \cdots \cup \beta_k$.

By Lemma, we know β lin. indep.

Also, $$\#\beta = \sum_{i=1}^{k} \#\beta_i \underset{\beta_i \text{ basis}}{=} \sum_{i=1}^{k} \gamma(\lambda_i) \underset{\text{assumption}}{=} \sum_{i=1}^{k} \mu(\lambda_i) \underset{\text{char. poly. splits.}}{=} \dim V.$$

Hence β is a basis for V consisting of eigenvectors of T.

i.e., T is diagonalizable. □